DOMINANT ENERGY CONDITION AND CAUSALITY FOR SKYRME-LIKE GENERALIZATIONS OF THE WAVE-MAP EQUATION

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ABSTRACT. It is shown in this note that a class of Lagrangian field theories closely related to the wave-map equation and the Skyrme model obeys the dominant energy condition, and hence by Hawking's theorem satisfies finite speed of propagation. The subject matter is a generalization of a recent result of Gibbons.

1. INTRODUCTION

Recently Gibbons showed [Gib03] that the Skyrme model obeys the dominant energy condition, and thus settling the problem of causality for that equation. In this note we will give a different proof of the same fact that easily generalizes to a class of Lagrangian field theories that includes, as special cases, the wave-map equation, the Skyrme model, and the Born-Infeld model.

Let (M, g) be an $m+1$ dimensional Lorentzian manifold, where sign convention is taken to be $(-, +, +, \cdots)$, and let (N, h) be an n dimensional Riemannian manifold. Let $\phi : M \to N$ be a C^1 map. Then the action of ϕ can be used to pull back the metric h onto M as a positive semi-definite quadratic form on TM, we write it as

$$\phi^* h(X, Y) = h(d\phi \cdot X, d\phi \cdot y)$$

where the left hand side is evaluated at a point $p \in M$ and the right hand side at the point $\phi(p) \in N$ for $X, Y \in T_pM$. Composing with the inverse metric g^{-1} we obtain the so-called *strain tensor* D^ϕ, a section of $T^1_1 M$:

$$D^\phi = g^{-1} \circ \phi^* h \ , \tag{1}$$

thus at every point p, D^ϕ is a linear transformation of T_pM. Now, if g were a Riemannian metric, then for a fixed basis of T_pM, the matrix (D^ϕ) is positive semi-definite. This is, unfortunately, no longer true in the Lorentzian case, and thus the eigenvalues of (D^ϕ) are in general complex.

Let $\{\lambda_1, \dots, \lambda_k\}$ denote the non-zero eigenvalues, counted with multiplicity, of (D^ϕ). Note that by elementary linear algebra, using that g is non-degenerate and h is positive definite, one easily sees that

$$k \leq rank(d\phi) \leq \min(m + 1, n) \ . \tag{2}$$

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Recall the elementary symmetric polynomials $s_j(\{\lambda_1,\ldots,\lambda_k\})$ given by

$$s_j(\{\lambda_1,\ldots,\lambda_k\}) = \sum_{1\leq\alpha_1<\alpha_2<\cdots<\alpha_j\leq k}\prod_{i=1}^{j}\lambda_{\alpha_i} \tag{3}$$

with $s_0 = 1$ and $s_j = 0$ for all $j > k$. Observe that for the $(m+1)\times(m+1)$ matrix (D^ϕ), the elementary symmetric polynomials correspond to the coefficients of the characteristic polynomial, and specifically $s_1 = tr(D^\phi)$ and $s_{m+1} = \det(D^\phi)$. By abuse of notation, we will write $s_j(D^\phi)$ when we mean the symmetric polynomials on the eigenvalues of (D^ϕ). Note that $s_j(D^\phi)$ is independent of a basis chosen for the vector space T_pM.

For a given class $\mathcal{A}$ of maps $\phi : M \to N$, we write

$$U_{\mathcal{A}} := \{v\in\mathbb{R}^{m+1} \mid v = (s_1,\ldots,s_{m+1})(D^\phi)\ ,\ \phi\in\mathcal{A}\}\ .$$

Definition 1. *For a given class $\mathcal{A}$, let $\mathcal{U}_{\mathcal{A}}\subset\mathbb{R}^{m+1}$ be an open set that contains $U_{\mathcal{A}}\cup\{0\}$. An* admissible function $F : \mathcal{U}_{\mathcal{A}}\to\mathbb{R}$ *for the class $\mathcal{A}$ is a sub-additive, concave function, that is C^1 on the interior of $\mathcal{U}_{\mathcal{A}}$ and continuous up to the boundary.*

Remark 2. *In the definitions above, it only suffices to include terms up to s_{m+1} in view of* (2)*. Also, observe that sub-additivity and concavity of F immediately implies that $F(0)\geq 0$.*

Definition 3. *A Lagrangian field theory for the class $\mathcal{A}$ of maps $\phi : M\to N$ is said to be a* generalized wave-map[1] *if the Lagrangian*

$$L = F(s_1(D^\phi), s_2(D^\phi),\ldots, s_{m+1}(D^\phi))$$

for an admissible F. Furthermore, we say that the generalized wave-map is defocusing *if the first partial derivatives of F are all non-negative, i.e. $\partial_i F(v)\geq 0$ $\forall i = 1,\ldots,m+1$ and $\forall v\in\mathcal{U}_{\mathcal{A}}$. The generalized wave-map is said to be* zeroed *if $F(0) = 0$. Also, we shall refer to a generalized wave-map for which $\partial_1 F$ is non-vanishing as* non-degenerate.

The author hopes that the reason behind the nomenclature will be evident after the proof of the dominant energy condition is developed. We first give some examples of generalized wave-maps:

- Observe that if L is a linear combination of the symmetric polynomials $L = \sum c_i s_i(D^\phi)$, than it is automatically a zeroed generalized wave-map. If in addition the coefficients c_i are all non-negative, then L is defocusing. In this case if $c_1 > 0$ then L is non-degenerate.
- Take (M, g) to be a static space-time, i.e. $M = \mathbb{R}\times\Sigma$ and $g = -\rho dt^2\oplus\gamma$ where ρ is a positive function on Σ and γ is a Riemannian metric on Σ. A static solution to the generalized wave-map is one for which $\nabla_t\phi = 0$. The static solution for $L = s_1$ gives rise to the harmonic map equation from $\Sigma\to N$, while for the case $n > m$, $L = \sqrt{s_m}$ (recall that $\dim M = m+1$), the equation becomes the minimal surface equation for the embedding of Σ into N. For the minimal surface equation we take $\mathcal{U}_{\mathcal{A}} = \bar{\mathbb{R}}_+^{m+1}$.
- In the Lorentzian case, $L = s_1$ is simply the wave-map equation. For $L = c_1 s_1 + c_2 s_2$ where $c_1, c_2 > 0$ are coupling constants, we recover the original Skyrme model if we take (N, h) to be $SU(2)$ with the bi-invariant metric.

[1] For the lack of a better name. Suggestions are welcome.

In particular, the Skyrme model is a defocusing, zeroed, non-degenerate, generalized wave-map in the terminology adopted in the present paper.

- Let $b > 0$ be a fixed large constant. We can restrict ϕ to only consider those maps such that the real parts of the eigenvalues of D^ϕ are greater than $-b$. Then letting
$$F = \sqrt{\det(b \cdot Id + D^\phi)} - \sqrt{\det(b \cdot Id)}$$
defined on $\mathcal{U}_\mathcal{A}$ being the set where $\det(b \cdot Id + D^\phi) \geq 0$, we get the zeroed, defocusing, non-degenerate, generalized wave-map also known as the Born-Infeld model.

Before stating the main theorem, we recall the statement of the dominant energy condition. Recall that the (covariant) stress-energy tensor $T \in \Gamma(T^0_2 M)$ for a Lagrangian field theory is given by a variational derivative for the Lagrangian *density* relative to the inverse metric,

$$T\sqrt{|\det g|} := \frac{\delta[L\sqrt{|\det g|}]}{\delta g^{-1}} = \left(\frac{\delta L}{\delta g^{-1}} - \frac{1}{2}Lg\right)\sqrt{|\det g|}\ . \tag{4}$$

Definition 4. *The stress-energy tensor T is said to obey the* dominant energy condition *at a point $p \in M$ if $\forall X \in T_pM$ such that $g(X, X) < 0$, the following two conditions are satisfied*

$$T(X, X) > 0 \tag{5a}$$
$$[T \circ g^{-1} \circ T](X, X) \leq 0 \tag{5b}$$

unless T vanishes identically.

Remark 5. *The definition is equivalent to the classical statements (see, e.g. section 4.3 in* [HE73] *or chapter 9 of* [Wal84]*) of the dominant energy condition. Observe that* (5b) *gives that the vector $g^{-1} \circ T \circ X$ is a causal vector for any time-like vector X, and* (5a) *gives that the vector $g^{-1} \circ T \circ X$ has opposite time-orientation as the time-like vector X.*

Now we state the main theorem

Theorem 6. *A defocusing generalized wave-map obeys the dominant energy condition.*

First we claim that it would suffice to prove the theorem for each s_i. The fellowing lemma is a general statement on a convexity property of Lagrangian field theories.

Lemma 7. *Let F be a sub-additive, concave function as in Definition 3. Let T_i denote the stress-energy tensor corresponding to the Lagrangian L_i. Assume that T_i obeys the dominant energy condition, or, equivalently, the vectors $Y_i = g^{-1} \circ T_i \circ X$ are all past-causal for any fixed future time-like X. Then $L = F(L_1, \ldots, L_{m+1})$ also obeys the dominant energy condition if L is defocusing.*

Proof. The stress-energy tensor T can be written, using (4), as

$$T = \sum_{i=1}^{m+1} \partial_i F \cdot \frac{\delta L_i}{\delta g^{-1}} - \frac{1}{2}Fg = \sum_{i=1}^{m+1} \partial_i F \cdot T_i - \frac{1}{2}(F - \sum_{i=1}^{m+1} \partial_i F \cdot L_i)g\ .$$

Now considering $g^{-1} \circ T \circ X$, the first term in the above expression contributes $\sum \partial_i F \cdot Y_i$. Since L is defocusing, this is a positive linear combination of past-causal vectors, and hence by elementary Minkowskian geometry, is still past-causal. For the second term, since $g^{-1} \circ g \circ X = X$, to show that it is also past-causal it suffices to show that

$$F \geq \sum_{i=1}^{m+1} \partial_i F \cdot L_i \ .$$

But this follows from the fact that F is concave and $F(0) \geq 0$. □

Unfortunately, it is immediately clear that the theorem may not be strong enough in certain cases for practical application. This is because the vanishing of T does not guarantee that the map ϕ is trivial. For example, using that $s_j = 0$ if $j > rank(d\phi)$, it is immediate that if locally around the point p, ϕ is one-dimensional, then for any metric g, $s_j(D^\phi) = 0$ if $j \geq 2$. On the other hand, this failure of the dominant energy condition arises from a degeneracy which forces the stress-energy tensor to be a null stress tensor in the language of Christodoulou [Chr00], which we can "normalize" away by taking L to be zeroed. We claim that this is the only possible failure.

Proposition 8. *For $L = s_i$, T obeys the dominant energy condition. Furthermore, $T = 0$ at a point p if and only if $i > rank(d\phi|_p)$.*

From this proposition one immediately sees the following energy bound for smooth solutions of the generalized wave-map equation.

Corallary 9. *If ϕ is the solution to a defocusing, non-degenerate, zeroed, generalized wave-map, and if $T = 0$ on a connected open domain $\mathcal{B}$ of M, then ϕ is constant on $\mathcal{B}$.*

By applying Hawking's energy conservation theorem (see section 4.3 in [HE73]) the above corollary implies that defocusing, non-degenerate, zeroed, generalized wave-maps have finite speed of propagation (also known as the domain of dependence condition).

In principle, if one has advanced knowledge on a lower bound to the rank of the map ϕ, one can also obtain analogous statements for degenerate cases. We leave such trivial generalizations to the reader.

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2. A formula for the stress-energy tensor and proof of the main proposition

In this section, we'll derive first derive a formula for the stress-energy tensor. We will begin by making a geometric observation and obtain, almost immediately, a simple tensorial formula for the Lagrangian. Taking the formal variational derivative of the Lagrangian leads to a tensorial expression for the stress-energy tensor, from which Proposition 8 follows via simple linear algebra.

Consider a real vector space V. Let A be a linear transformation on V. Then A naturally extends to a linear transformation, which we denote $A^{\sharp j}$, on $\Lambda^j(V)$, the space of alternating j-vectors over V. A bit of basic linear algebra (perhaps by extending V to $V \otimes_{\mathbb{R}} \mathbb{C}$ and taking a basis of eigenvectors) shows that $s_j(A)$ is

proportional to $tr_{\Lambda^j(V)} A^{\sharp j}$. Now, letting $V = T_pM$ and $A = D^\phi = g^{-1} \circ \phi^* h$, we observe that

$$(D^\phi)^{\sharp j} = (g^{-1})^{\sharp j} \circ \phi^*(h^{\sharp j}) \ ,$$

or, to put it in words, $(D^\phi)^{\sharp j}$ is obtained from first taking the induced metric $h^{\sharp j}$ on alternating j-vectors in $T_{\phi(p)}N$, pulling it back via ϕ, and composing it with the induced metric $(g^{-1})^{\sharp j}$ for the alternating j-forms. In index notation, this can be written as

$$[(D^\phi)^{\sharp j}]^{b_1\ldots b_j}_{a_1\ldots a_j} = g^{b_1c_1}\cdots g^{b_jc_j}(\phi^*h)_{a_1[c_1|}(\phi^*h)_{a_2|c_2|}\cdots(\phi^*h)_{a_{j-1}|c_{j-1}|}(\phi^*h)_{a_j|c_j]}$$

where the bracket notation in the indices denotes full anti-symmetrization of the $\{c_1, \ldots, c_j\}$ indices. For a Lagrangian proportional to an s_j, we can assume

$$L = [(D^\phi)^{\sharp j}]^{b_1\ldots b_j}_{a_1\ldots a_j} = g^{a_1[c_1|}\cdots g^{a_j|c_j]}(\phi^*h)_{a_1c_1}\cdots(\phi^*h)_{a_jc_j} \ . \tag{6}$$

It is simple to check, using $(D^\phi) = diag(-1, 1, 1, \ldots)$ that the above expression has the correct sign: that L defined thus is a positive multiple of s_j.

One can also arrive at (6) purely from a linear algebra point of view. Let p_j be the power sum

$$p_j(\{\lambda_1, \ldots, \lambda_k\}) = \sum_{i=1}^k \lambda_i^j \ .$$

Recall that we have Newton's identity

$$j \cdot s_j = \sum_{i=1}^j (-1)^{i-1} e_{j-i} p_i$$

which allows us to express s_j as a rational polynomial in p_i's. Now, by definition, it is clear that

$$p_j(D^\phi) = tr[(D^\phi)^j]$$

where $(D^\phi)^j$ is the j-fold composition of D^ϕ. It is easy to check then, for some E

$$s_j = g^{a_1b_1}\cdots g^{a_jb_j} E^{c_1\ldots c_j}_{b_1\ldots b_j}(\phi^*h)_{a_1c_1}\cdots(\phi^*h)_{a_jc_j} \ .$$

Newton's identity reduces to a generating condition for E based on the Kronecker δ symbols,

$$E^c_b = \delta^c_b \ ,$$

$$jE^{c_1\ldots c_j}_{b_1\ldots b_j} = \sum_{i=1}^j (-1)^{i-1} E^{c_1\ldots c_{j-i}}_{b_1\ldots b_{j-i}} \delta^{c_{j-i}}_{b_{j-i+1}} \delta^{c_{j-i+1}}_{b_{j-i+2}} \cdots \delta^{c_{j-i+1}}_{b_j} \ .$$

A direct computation which we omit here shows that then in fact the invariant $E^{c_1\ldots c_j}_{b_1\ldots b_j}$ is a positive rational multiple of the generalized Kronecker symbol $\delta^{c_1\ldots c_j}_{b_1\ldots b_k}$, from which we recover (6).

Now, the object we are interested in, given a time-like vector X, is the one-form $T(X, \cdot)$. Since T is tensorial, we can assume X has unit length. Fix some j, let the Lagrangian be proportional to s_j as given by (6). By the symmetry property, we can write $T(X, \cdot)$ in index notation:

$$T_{ab}X^b = jX^{[b|}g^{a_2|c_2|}\cdots g^{a_j|c_j]}(\phi^*h)_{ab}\cdots(\phi^*h)_{a_jc_j} - \frac{1}{2}g_{ab}X^bL \tag{7}$$

Proof of Proposition 8. Consider a orthonormal basis for T_pM relative to g. Since we assumed X unit, let $e_0 = X$ and $\{e_i\}_{1\leq i\leq m}$ are all space-like. We can take $j \leq m+1$ as otherwise T is identically 0. Then we notice that a basis for $\Lambda^j(T_pM)$ is given by

$$\{e_0 \wedge e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_{j-1}}\}_{1\leq\alpha_1<\cdots<\alpha_{j-1}\leq m} \cup \{e_{\alpha_1} \wedge \cdots \wedge e_{\alpha_j}\}_{1\leq\alpha_1<\cdots<\alpha_j\leq m} \ .$$

We write the first set as $\Lambda^j_\perp$ and the second set as $\Lambda^j_\parallel$. Using the normalization that $v \wedge w = v \otimes w - w \otimes v$, we find that each of the element in $\Lambda^j_\perp$ has norm $-j!$ while the elements in $\Lambda^j_\parallel$ has norm $j!$.

To show that $T(X,X) > 0$ generically, we observe that under the expansion (7), the first term corresponds to

$$\sum_{\omega\in\Lambda^j_\perp} \phi^*(h^{\sharp j})(\omega,\omega) \ ,$$

while the second term corresponds to

$$\frac{1}{2}\left(-\sum_{\omega\in\Lambda^j_\perp} \phi^*(h^{\sharp j})(\omega,\omega) + \sum_{\omega\in\Lambda^j_\parallel} \phi^*(h^{\sharp j})(\omega,\omega)\right) \ .$$

So summing them gives

$$\frac{1}{2}\left(\sum_{\omega\in\Lambda^j_\perp} \phi^*(h^{\sharp j})(\omega,\omega) + \sum_{\omega\in\Lambda^j_\parallel} \phi^*(h^{\sharp j})(\omega,\omega)\right)$$

which is non-negative by the fact that $\phi^*(h^{\sharp j})$ is a positive semi-definite quadratic form on $\Lambda^j(T_pM)$. Furthermore, observe that since $\Lambda^j_\parallel \cup \Lambda^j_\perp$ is a basis, its push-forward $\phi_*\Lambda^j_\parallel \cup \phi_*\Lambda^j_\perp$ spans $\Lambda^j(\phi_*T^M_p) \subset \Lambda^j(T_{\phi(p)}N)$. Thus by the fact that h (and hence the induced metric $h^{\sharp j}$) is positive definite, we conclude that $\Lambda^j(\phi_*T^M_p) = \{0\}$, which proves the assertion that T vanishes only when $j > rank(d\phi)$.

To show (5b), we observe that

$$X^aT_{ac}g^{cd}T_{db}X^b = -T(X,X)^2 + \sum_{i=1}^m T(X,e_i)^2 \ .$$

The first thing to note is that $T(X,e_i)$ does not have any contribution from the second term in (7). For the first term, a quick computation shows that $T(X,ei)$ corresponds to

$$\sum_{\eta\in\Lambda^{j-1}_\parallel} \phi^*(h^{\sharp j})(e_0\wedge\eta, e_i\wedge\eta)$$

so

$$
\begin{aligned}
|\sum_{i=1}^{m} T(X,e_i)^2| &\le (\sum |T(X,e_i)|)^2 \\
&\le (\sum_{i=1}^{m} \sum_{\eta\in\Lambda_{\|}^{j-1}} |\phi^*(h^{\sharp j})(e_0\wedge\eta, e_i\wedge\eta)|)^2 \\
&\le \frac{1}{4}(\sum_{\eta\in\Lambda_{\|}^{j-1}} \phi^*(h^{\sharp j})(e_0\wedge\eta, e_0\wedge\eta) + \sum_{i=1}^{m} \phi^*(h^{\sharp j})(e_i\wedge\eta, e_i\wedge\eta))^2 \\
&= \frac{1}{4}(\sum_{\eta\in\Lambda_{\|}^{j-1}} \sum_{i=0}^{m} \phi^*(h^{\sharp j})(e_i\wedge\eta, e_i\wedge\eta))^2 \\
&= T(X,X)^2
\end{aligned}
$$

And therefore (5b) is satisfied. □

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